

Generation Permutation Symmetry and the Quark Mixing Matrix

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Abstract

We propose a new discrete symmetry in the generation space of the fundamental fermions, consistent with the observed fermion mass spectrum. In the case of the quarks, the symmetry leads to the unique prediction of a flat CKM matrix at high energy. We explore the possibility that evolution due to quantum corrections leads to the observed hierarchical form of the CKM matrix at low energies.

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The problem of the origin of the masses and the mixing angles of the fundamental fermions must surely be amongst the most urgent in particle physics today. Even accepting the standard mechanism for fermion mass generation through Yukawa couplings to one or more non-zero Higgs fields, the reason for the existence of three fermion generations together with the explanation for the observed pattern of the individual masses and mixing angles remains mysterious. One possible way forward is to gain experience by constructing and analysing a wide variety of plausibly motivated candidate mass matrices (or ansätze) in the hope that something convincing will eventually emerge. Amongst the best known and perhaps the most thoroughly analysed such ansatz is that due to Fritzsch [1]. The present proposal has more in common with the approach pioneered by Harari et al. [2].

In this paper we motivate and analyse a new ansatz for the fermion mass matrices, which we believe has unique a priori appeal by virtue of the principles underlying its construction. Our proposal owes something to the straightforward and oft-repeated observation that the fermion generations are in some (yet to be defined) sense duplicate copies one of the other. That is to say that, in spite of the large mass differences observed from generation to generation, it is natural to assume that the three generations exist fundamentally on an equal footing. In constructing our ansatz, we take this notion seriously and insist that, at the most fundamental level, there be no physical basis for preferring one generation over another, ie. in the Lagrangian the assignment of the generation labels ($i = 1-3$) must be entirely arbitrary. Such a demanding requirement has much of the character of established invariance principles in physics, and naturally puts very severe constraints on the form that the mass matrices can take. Indeed these constraints are so severe that they can often appear at first sight to be in conflict with the experimental facts. We show in this paper, however, that this is not necessarily the case. The indisputable a priori appeal of the above idea, taken together with the uniqueness and economy of its implementation, have provided much of the motivation to pursue this analysis.

We begin by noting that a principle of the sort outlined above is trivially satisfied by the charged-current weak interaction in any weak basis, as a consequence of the universality of the weak interaction. On the other hand, the evident large mass differences observed, from generation to generation, tell us that the Yukawa couplings in the physical basis, are quite definitely not universal. At this point, the *only* solution that we can see, consistent with the principle we have expounded above, requires that we postulate that in some weak basis the Yukawa couplings for a given fermion species exhibit an invariance under *permutations* of the generation indices. A candidate mass

matrix fulfilling our requirement, which is also hermitian is:

$$m = \begin{pmatrix} a & b & b^* \\ b^* & a & b \\ b & b^* & a \end{pmatrix} \quad (1)$$

where a is real and b is complex. Note that the diagonal mass terms are all identical (they are all equal to a) and that the off-diagonal (weak-generation-changing) amplitudes for the ‘clockwise’ transistions ($1 \rightarrow 2$, $2 \rightarrow 3$ and $3 \rightarrow 1$) are also all identical (they are all equal to b) and the amplitudes for the ‘anticlockwise’ transistions ($1 \rightarrow 3$, $3 \rightarrow 2$, and $2 \rightarrow 1$) are all equal to b^* , so that no generation is preferred. A matrix of this form is sometimes referred to as a circulant [3]. It might be argued that the mass matrices are unlikely to be hermitian and that a general circulant matrix with a complex and with unrelated complex numbers b and c representing different amplitudes for the clockwise and anticlockwise transistions, would also satisfy our requirement. Nothing is to be gained, however, by postulating this general form since, on taking the hermitian square (mm^\dagger), we immediately recover the form eq.(1), and, as is well known, only the hermitian square of the mass matrix can influence the measured masses and mixing angles.

Suppose that we postulate a matrix of the above form for the hermitian square of the mass matrix for the charged leptons. The observed mass spectrum can be reproduced by setting:

$$\begin{aligned} a &= (\tau/3) + (\mu/3) + (e/3) \\ b &= (\tau/3) \omega_1 + (\mu/3) \omega_2 + (e/3) \omega_3 \end{aligned} \quad (2)$$

where τ , μ and e represent the masses-squared of the τ -lepton, muon and electron respectively, and the ω_i , $i = 1-3$ are the usual complex cube-roots of unity. In this form, in the rank-1 limit ($\mu, e \rightarrow 0$) the above matrix reproduces the matrix proposed by Harari et al. [2]. The form of eq.(2) follows from the general result that the spectrum of the eigenvalues of a circulant matrix is given by the (discrete) Fourier transform of its trailing diagonal. The eigenvectors of a matrix of the form eq.(1) are: $(1,1,1)$, $(1,\omega_2,\omega_3)$, $(1,\omega_3,\omega_2)$. These are of course just the momentum eigenstates for a three-point one-dimensional lattice satisfying periodic boundary conditions. An operator of the form eq.(1) (with b real and negative) was employed by Feynman [4] to describe the low lying energy states of the tri-phenyl-cyclo-propanyl ion. We consider it very significant that the matrix operator defined by eq.(1) and eq.(2) has so much in common with the simple derivative operators representing the ordinary kinetic terms in the Lagrangian, which as a consequence of translational invariance may also be represented by (infinite) circulant matrices. It might also be worth noting that the form eq.(1) may equivalently be regarded as the 3×3 generalisation of the phenomenologically successful 2×2

effective-theory [5] used to describe the properties of the neutral kaon system, prior to the discovery of CP violation.

Turning now to the quark mass matrices one might be tempted to postulate mass matrices of the form eq.(1), but with different parameters a and b , chosen in analogy with the case of the leptons above, so as to reproduce the observed mass spectrum for the up-type and down-type quarks respectively. But matrices of the form eq.(1) commute with each other for all values of a and b , so that the mass matrices for the up-type and down-type quarks would be simultaneously diagonalisable and the quark mixing (CKM [6]) matrix would then be the identity (or a trivial permutation matrix), in clear disagreement with experiment.

With these considerations in mind, we have investigated mass matrices of the somewhat more general form:

$$m = \begin{pmatrix} a & be^{i\phi_3} & b^*e^{-i\phi_2} \\ b^*e^{-i\phi_3} & a & be^{i\phi_1} \\ be^{i\phi_2} & b^*e^{-i\phi_1} & a \end{pmatrix} \quad (3)$$

with a and b still given by eq.(2) and with $\phi_1 + \phi_2 + \phi_3 = 0$, so that the mass eigenvalues are unchanged. In eq.(3) the off-diagonal amplitudes are equal in magnitude but differ in phase, so that the matrix eq.(3) does not commute with the matrix eq.(1), nor does it commute with matrices of the form eq.(3) with different values for the phases. The eigenvectors of a matrix of the form eq.(3) are: $(1, e^{-i\phi_3}, e^{i\phi_2})$, $(1, \omega_2 e^{-i\phi_3}, \omega_3 e^{i\phi_2})$, $(1, \omega_3 e^{-i\phi_3}, \omega_2 e^{i\phi_2})$. If we postulate matrices of the form eq.(3) for (the hermitian squares of) the mass matrices for the up-type and down-type quarks, and construct unitary matrices U and D comprising the respective mass-ordered normalised eigenvectors, we find that the CKM matrix ($V = U^\dagger D$) may then itself be written as a circulant:

$$V = \begin{pmatrix} p & q & r \\ r & p & q \\ q & r & p \end{pmatrix}. \quad (4)$$

Observables depend only on the phase differences ($\Delta\phi_i$) between the corresponding amplitudes in the up-type and down-type mass matrices:

$$\begin{aligned} |p|^2 &= (3 + 2\text{Re}S)/9 \\ |q|^2 &= (3 - \text{Re}S + \sqrt{3}\text{Im}S)/9 \\ |r|^2 &= (3 - \text{Re}S - \sqrt{3}\text{Im}S)/9 \end{aligned} \quad (5)$$

with $S = e^{i\Delta\phi_1} + e^{i\Delta\phi_2} + e^{i\Delta\phi_3}$. The convention independent CP violation parameter J_{CP} [7] is given by:

$$J_{CP} = \frac{1}{27}\text{Im}(e^{i(\Delta\phi_2 - \Delta\phi_1)} + e^{i(\Delta\phi_3 - \Delta\phi_2)} + e^{i(\Delta\phi_1 - \Delta\phi_3)}) \quad (6)$$

For example, if $\Delta\phi_1 = 0^\circ$ and $\Delta\phi_2 = 60^\circ$ (and hence $\Delta\phi_3 = -60^\circ$) then $S = 2$ and $|p| = \sqrt{7}/3 \simeq 0.882$, $|q| = |r| = 1/3 \simeq 0.333$ and $J_{CP} = 1/(18\sqrt{3}) \simeq 0.032$. We see no way to justify such a choice of phases however.

At this point, we return again to the similarity we noted above, between the operator eq.(1) and the simple derivative operators representing the ordinary kinetic terms in the Lagrangian. Building on this observation, we now note that a close analogy exists between the operator eq.(3) and (the hermitian square of) a full gauge-covariant kinetic operator. The phases ϕ_i ($i = 1-3$) play a role here analogous to that of the gauge potential. The freedom to change the absolute phases using any (common) arbitrary diagonal matrix of phase factors, is analogous to local gauge invariance. A gauge-field configuration corresponding to a constant field-strength (ie. a uniform field) is of particular interest to us here, because a uniform field is manifestly translationally invariant. We note that even in the case of a uniform field, the inherent translational invariance cannot be explicit in all of the components of the gauge potential at once, after a choice of gauge has been made. In the same way if we set:

$$\Delta\phi_2 - \Delta\phi_1 = \Delta\phi_3 - \Delta\phi_2 = \Delta\phi_1 - \Delta\phi_3 \quad (7)$$

corresponding to a uniform field (in the discrete generation space), then it must be that no generation is preferred, even though the up-type and the down-type mass matrices clearly cannot both be circulant. As far as observables are concerned, this last requirement eq.(7) (together with the requirement $\Delta\phi_1 + \Delta\phi_2 + \Delta\phi_3 = 0$, above) completely specifies our ansatz (eg. $\Delta\phi_1 = 0^\circ$, $\Delta\phi_2 = \pm 120^\circ$, $\Delta\phi_3 = \mp 120^\circ$), up to the sign of J_{CP} . The CKM matrix is flat in this case, ie. all elements have equal modulus $|p| = |q| = |r| = 1/\sqrt{3} \simeq 0.577$, and J_{CP} is extremal, ie. $|J_{CP}| = 1/(6\sqrt{3}) \simeq 0.096$ [7].

If the above matrices are relevant at all, they are relevant only at very high energy, eg. unification (GUT) energies, and have to be evolved down to the electro-weak (EW) scale in order to be compared with experiment. The leading-order evolution equations [8] for the quark Yukawa matrices in the Standard Model (SM) can be written (neglecting the influence of the charged leptons):

$$\begin{aligned} \dot{\alpha}_u &= \frac{3}{2}\alpha_u^2 - \frac{3}{4}(\alpha_u\alpha_d + \alpha_d\alpha_u) + 3\text{Tr}(\alpha_u + \alpha_d)\alpha_u - 8\alpha_3\alpha_u - \frac{9}{4}\alpha_2\alpha_u - \frac{17}{20}\alpha_1\alpha_u \\ \dot{\alpha}_d &= \frac{3}{2}\alpha_d^2 - \frac{3}{4}(\alpha_u\alpha_d + \alpha_d\alpha_u) + 3\text{Tr}(\alpha_u + \alpha_d)\alpha_d - 8\alpha_3\alpha_d - \frac{9}{4}\alpha_2\alpha_d - \frac{5}{20}\alpha_1\alpha_d \quad (8) \\ \dot{\alpha}_3 &= -7\alpha_3^2 \quad \dot{\alpha}_2 = -\frac{19}{6}\alpha_2^2 \quad \dot{\alpha}_1 = \frac{41}{10}\alpha_1^2 \end{aligned}$$

where Tr denotes the matrix trace, the dot denotes differentiation with respect to $T = (1/2\pi) \ln(E/E_0)$ and E/E_0 is the running energy scale, expressed as a fraction of the starting energy. The hermitian squares of the up-type and the down-type Yukawa matrices are represented by α_u and α_d respectively, where a factor of $1/4\pi$ has been

incorporated in the definition of α_u and α_d to simplify the form of the evolution equations, in analogy with the case of the gauge couplings. The corresponding equations for the gauge couplings (α_i , $i = 1-3$) are included for completeness.

There has been much progress in understanding the effects of evolution analytically [9], but for simplicity the results presented here are based on a straightforward numerical integration of eq.(8), employing an appropriate (variable) stepsize. Suitable starting values for the gauge couplings are taken from the fits of Amaldi et al. [10]. For a given set of starting values for the Yukawa couplings, we calculate the quark mass spectrum and the CKM matrix at the lower energy scale. There is considerable freedom in choosing starting values for the Yukawa couplings consistent with the observed mass spectrum at low energies due (in large part) to the well known quasi-fixed-point [11], implicit in the evolution equations, which tends to focus the top Yukawa coupling towards its fixed-point value at low energies, independent of its starting value. In spite of this, we find that assuming *perturbative* starting values for the individual Yukawa couplings (ie. $\alpha_u, \alpha_d \lesssim 1$), chosen to reproduce the observed quark mass spectrum, the predicted evolution is always too slow to yield a realistic CKM matrix at low energies. Evolving down over a reasonable range in T (the GUT scale and the EW scale are about five units apart in T) the CKM matrix remains approximately flat; that is to say, all elements remain close to their starting value, $|V_{ij}| \simeq 1/\sqrt{3} \simeq 0.577$, to within deviations at the level of 20% or less.

However, with recent experimental results from LEP and from the Tevatron tending to favour large values for the top mass [12], it is becoming increasingly clear that the Yukawa couplings may very well be *non-perturbative* at high energy. Whilst we do not expect perturbative evolution equations to be quantitatively valid in a non-perturbative regime, we have done what we can to investigate this possibility, by applying eq.(8) also in the case that the Yukawa couplings assume non-perturbative values (ie. $\alpha_u, \alpha_d \gtrsim 1$). As one might expect, with larger starting values for the Yukawa couplings, the evolution proceeds more rapidly. The observed quark mass spectrum at low energy, can still be correctly reproduced, thanks to the quasi-fixed-point. We now find, however, that the CKM matrix, although starting out absolutely flat, rapidly develops a significant hierarchy which, for sufficiently large starting values for the Yukawa couplings, is not-at-all unlike the familiar hierarchy [13] of CKM amplitudes observed experimentally. That said, we have not succeeded in finding any one complete set of starting values which reproduces the quark mass spectrum and the CKM matrix simultaneously in every detail, and in view of the strict inapplicability of eq.(8) in the non-perturbative domain, neither should we expect to, even in the case that our ansatz was perfectly correct. Instead we give here a sample set of starting values that can be seen to reproduce most of the quark masses correctly, together with the main features of the

CKM matrix. The input values for the (diagonalised) Yukawa couplings at high energy are: $\alpha_u = (6.0 \times 10^{-2}, 2.0 \times 10^9, 7.0 \times 10^{11})$, $\alpha_d = (1.5 \times 10^{-1}, 5.0 \times 10^0, 4.5 \times 10^1)$ leading to $\alpha_u = (4.4 \times 10^{-11}, 8.3 \times 10^{-2}, 8.8 \times 10^{-2})$, $\alpha_d = (2.8 \times 10^{-10}, 6.0 \times 10^{-8}, 6.8 \times 10^{-5})$ at the EW scale ($\Delta T = -5$). The evolved CKM matrix is as follows (only the moduli of the elements are given here; phases are of course convention dependent):

$$V = \begin{pmatrix} 0.975 & 0.222 & 0.011 \\ 0.222 & 0.974 & 0.047 \\ 0.012 & 0.046 & 0.999 \end{pmatrix} \quad (9)$$

with $|J_{CP}| = 1.06 \times 10^{-4}$. The result eq.(9) bears a striking resemblance to the experimentally observed CKM matrix and suggests to us that it is evolution (albeit non-perturbative and presently incalculable) which is responsible for the observed hierarchy in the CKM matrix at low energy. Whilst results obtained by applying perturbative equations in a non-perturbative domain are unsatisfactory, in that they clearly cannot be used to falsify any hypothesis at all, we maintain that they do serve a useful purpose here as an illustration of existing possibilities. The problem of non-perturbative evolution may not be forever intractable: exact non-perturbative evolution equations for coupling constants in pure gauge theories have already been discussed in the literature [14]. Certainly it cannot be said that this ansatz is ruled out by experiment. On the contrary, if the trends we see applying leading-order perturbative evolution equations are at all representative of the effects of complete non-perturbative evolution, then all the indications are that we are on the right track.

In conclusion, in spite of the difficulties we have emphasised, we find the apparently natural emergence of a CKM-like hierarchy entirely within the SM framework very impressive. The matrix operators we have proposed come as close as one might hope to generalising (to the discrete generation space) the continuum gauge-covariant operators already present in the SM Lagrangian. One might even speculate that it is some analogue of the pure-gauge kinetic term, constructed from the relevant invariants [7], which (classically extremised) accounts for the hierarchy of quark masses. At the very least, we believe that we have demonstrated that this simple and appealing ansatz merits further investigation.

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